Basics about Wavelets

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PART THREE

Wavelets

On the menu ...

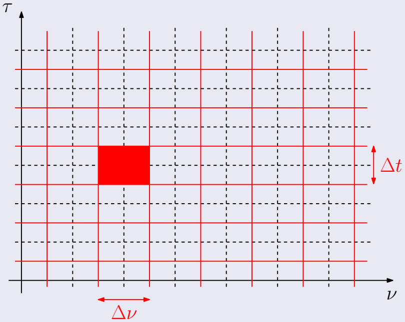
- Reminder about STFT limitations
- Families of wavelets and adaptive time-frequency plane
- Wavelet transform : definition and properties
- Multi-Resolution Analysis (MRA)
- 2D extensions
- Applications (approximation, denoising, compression)

Previously ...

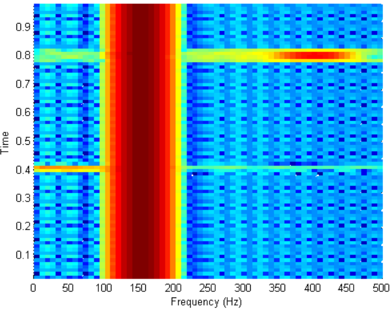
STFT

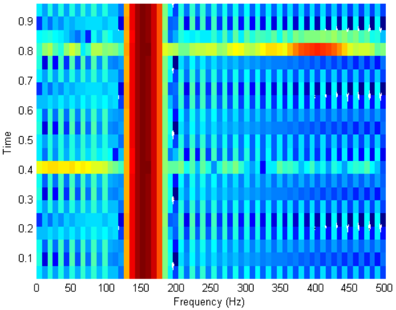
$$S_f(\nu,\tau) = \int_{-\infty}^{+\infty} w(t-\tau) f(t) e^{-\jmath 2\pi\nu t} dt$$

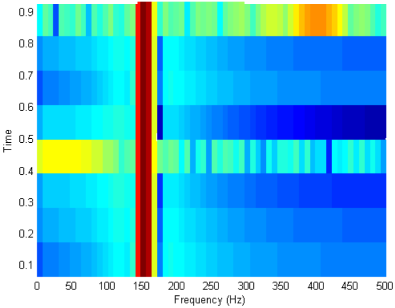
Prescribed time-frequency plane tiling



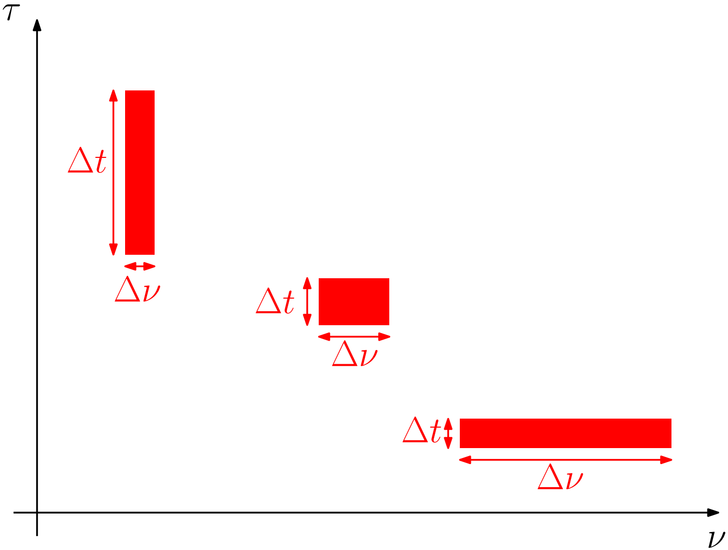
STFT limitations







Adaptative time-frequency plane



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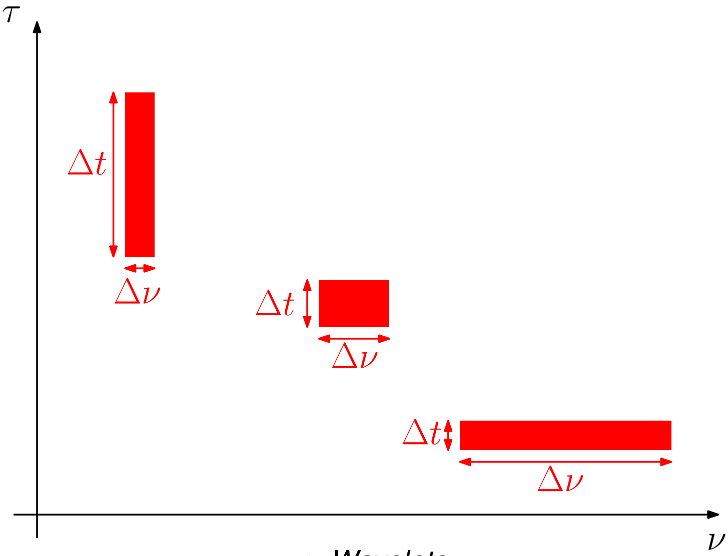

$\Rightarrow$ *Wavelets*

A few history ...

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- followers . . . : S.Mallat, I.Daubechies, R.Coiffman, A.Cohen, . . .

Before with Fourier

$$\hat{f}(\nu) = \int_{-\infty}^{+\infty} f(t) e^{-\jmath 2\pi\nu t} dt = \langle f, e^{\jmath 2\pi\nu t} \rangle$$

We project the signal f on the family of functions $\{e^{\jmath 2\pi\nu t}\}$.

It is this family which fixes the time-frequency plane tiling.

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Question : Is it possible to find a family of function in order to get the desired tiling ?

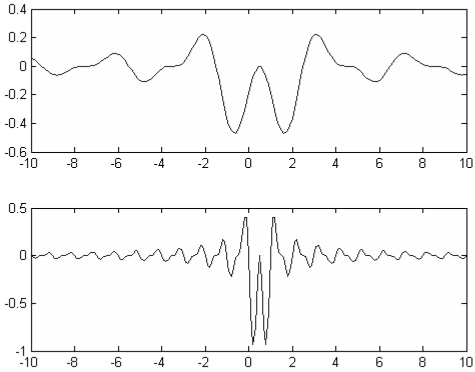
Wavelet family

We choose a "mother" wavelet $\psi(t)$ such that $\int_{-\infty}^{\infty} \psi(t) dt = 0$ (zero mean) and $\|\psi\|_{L^2} = 1$

We built a wavelet family by dilation/contraction (parameter $a \in \mathbb{R}^+$) and translation (parameter $b \in \mathbb{R}$) of the mother wavelet :

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi \left(\frac{t-b}{a} \right)$$

Note : $\|\psi_{a,b}\|_{L^2} = 1$



Continuous Wavelet Transform

CWT

$$W_f(a,b) = \langle f, \psi_{a,b} \rangle = \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{a}} \psi^* \left(\frac{t-b}{a} \right) dt$$

Admissibility condition : if $C_\psi = \int_0^{+\infty} \frac{|\hat{\psi}(\nu)|^2}{\nu} d\nu < +\infty$ then

Inverse transform

$$f(t) = \frac{1}{C_\psi} \int_0^{+\infty} \int_{-\infty}^{+\infty} W_f(a,b) \frac{1}{\sqrt{a}} \psi \left(\frac{t-b}{a} \right) db \frac{da}{a^2}$$

First properties

Energy conservation

$$\int_{-\infty}^{+\infty} |f(t)|^2 dt = \frac{1}{C_\psi} \int_0^{+\infty} \int_{-\infty}^{+\infty} |W_f(a,b)|^2 db \frac{da}{a^2}$$

CWT is a filtering process !

If we write

$$\bar{\psi}_a(t) = \frac{1}{\sqrt{a}} \psi^* \left(\frac{-t}{a} \right)$$

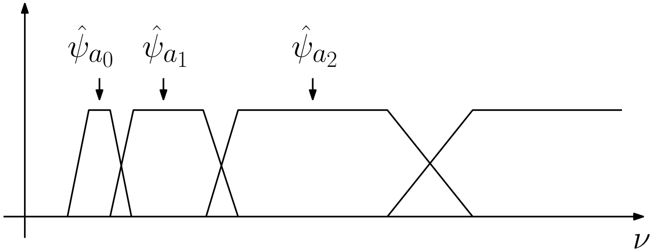
then

$$W_f(a,b) = f \star \bar{\psi}_a(b)$$

Note : $\widehat{\bar{\psi}_a}(\nu) = \sqrt{a}\widehat{\psi^*}(a\nu)$.

Interpretation

The wavelet transform is a passband filtering !



Consequence : we cannot get the zero frequency $\longrightarrow$ the missing part is obtained with the **scaling function.**

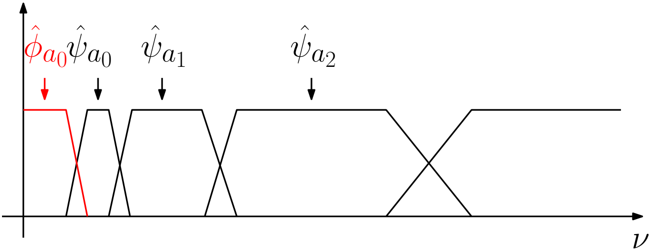
Scaling function 1/2

We know $W_f(a,b)$ for $a < a_0 \longrightarrow$ and we need to know the information corresponding to $a \geqslant a_0$ in order to perfectly rebuild f. To do this we use the scaling function which is defined from its Fourier transform and the FT of the wavelet :

$$|\hat{\phi}(\nu)|^2 = \int_{\nu}^{+\infty} |\hat{\psi}(\xi)|^2 \frac{d\xi}{\xi}$$

We can also write $\bar{\phi}_a(t) = \frac{1}{\sqrt{a}}\phi^*\left(\frac{-t}{a}\right)$ thus the missing information is captured by a lowpass filtering :

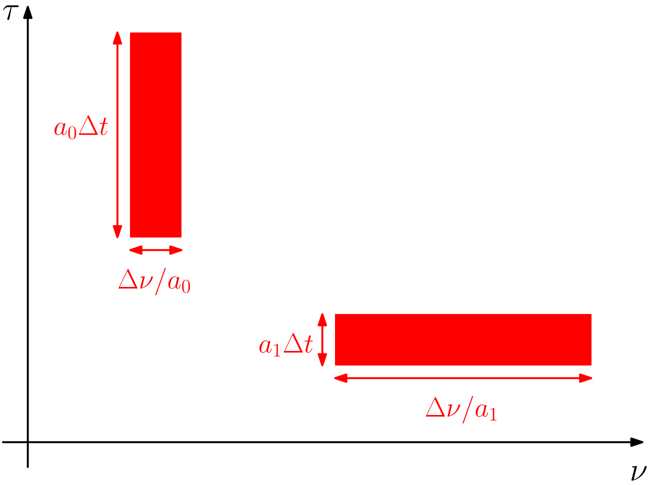
$$L_f(a,b) = \left\langle f(t), \frac{1}{\sqrt{a}}\phi\left(\frac{t-u}{a}\right)\right\rangle = f \star \bar{\phi}_a(u)$$



The reconstruction is obtained by :

$$f(t) = \frac{1}{C_\psi} \int_0^{a_0} W_f(a,.) \star \psi_a(t) \frac{da}{a^2} + \frac{1}{C_\psi a_0} L_f(a,.) \star \phi_{a_0}(t)$$

Time-frequency plane tiling



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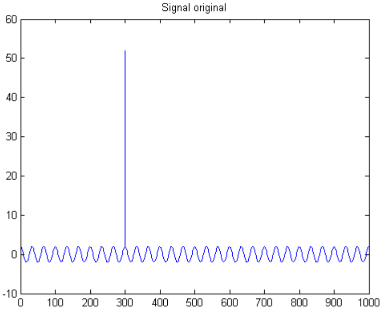
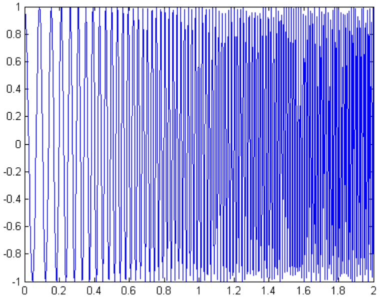
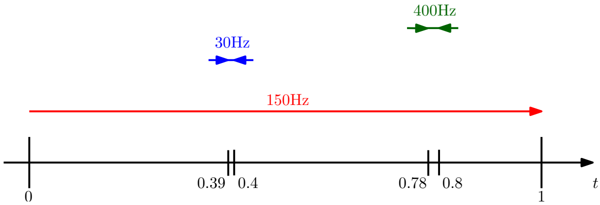
- Scaling function : $\phi_J[n] = \frac{1}{\sqrt{d^J}}\phi\left(\frac{n}{d^J}\right)$ where $a_0 = d^J$.
- Low frequencies are obtained by : $L_f[n, d^J] = \sum_{m=0}^{N-1} f[m]\phi_J^*[m-n] = f \circledast \bar{\phi}_J[n]$

Some examples 1/4

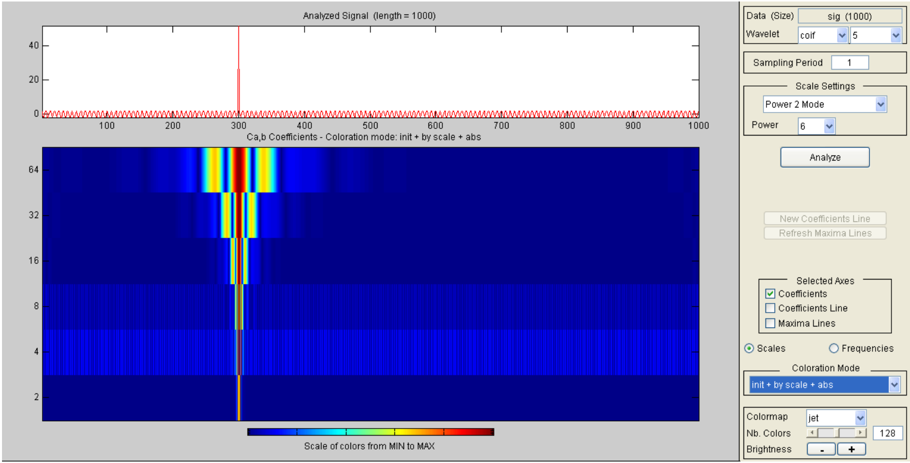
We use the same signals as in the Fourier slides :







Some examples 2/4



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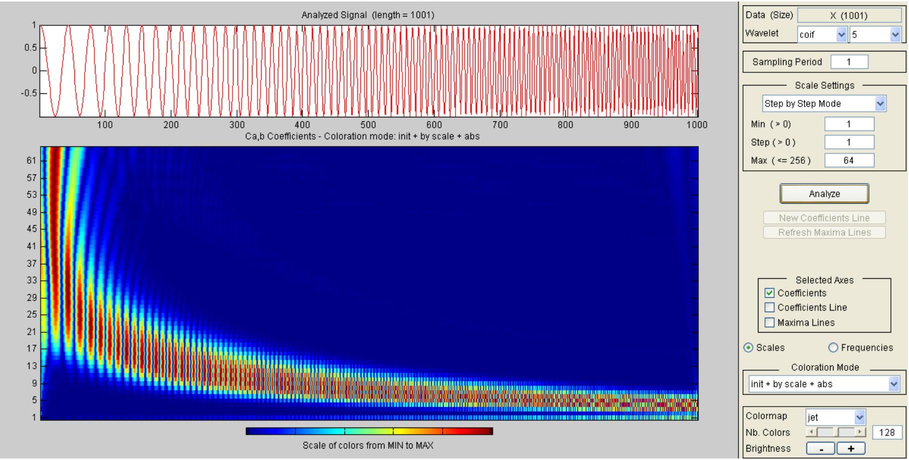
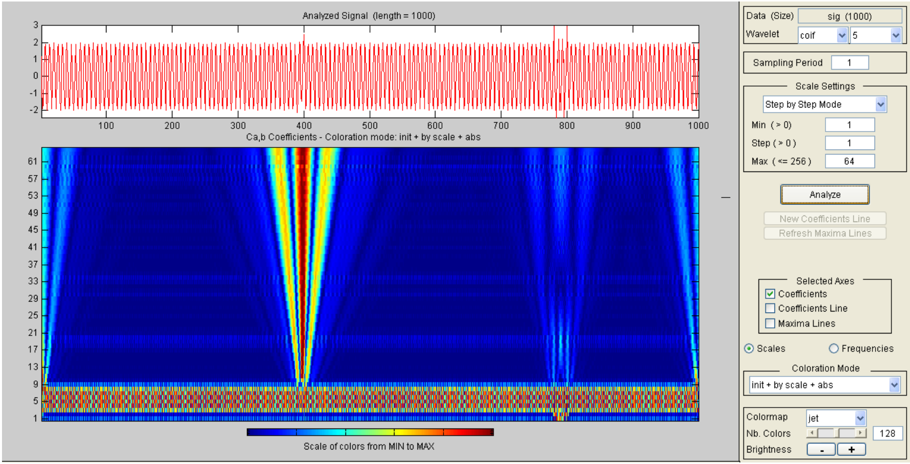
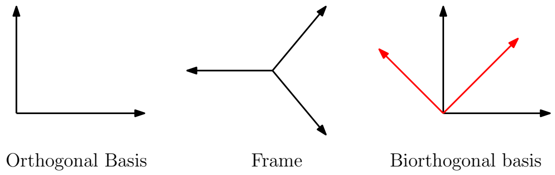


Some examples 4/4



Notion of Bases et Frames

In geometry



Orthogonal bases

$\{e_n\}$ is an orthogonal basis if $\langle e_n, e_m \rangle = 0$ if $m \neq n$.

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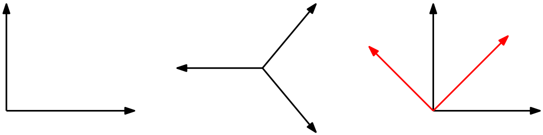

Orthogonal Basis   Frame   Biorthogonal basis

Frames

$\{e_n\}$ is a frame if $\exists A, B > 0$ such that
$A\|f\|_{L^2}^2 \leqslant \sum_{n\in\Gamma} |\langle f, e_n\rangle|^2 \leqslant B\|f\|_{L^2}^2$.
The reconstruction is obtained by using the pseudo-inverse operator using the dual frame $\{\tilde{e}_n\}$.
If $A = B$ then we have a **tight** frame

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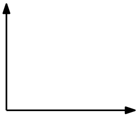
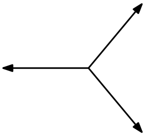
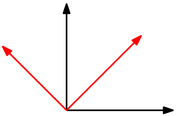


Orthogonal Basis

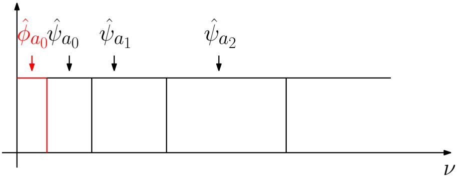


Frame



Biorthogonal basis

Orthogonal basis



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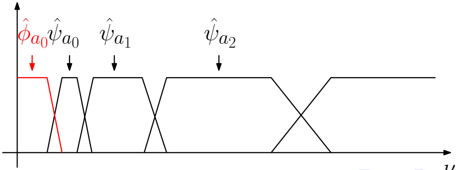

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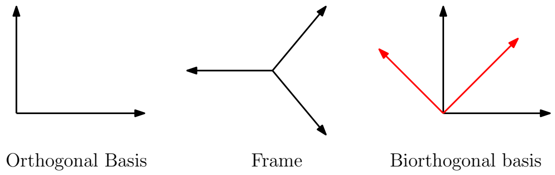


Biorthogonal basis

Let two wavelet families $\{\psi_{j,n}\}$ and $\{\tilde{\psi}_{j,n}\}$, they are said biorthogonal if

$$\langle \psi_{j,n}, \tilde{\psi}_{j',n'} \rangle = \delta[n-n']\delta[j-j'] \qquad \forall (j,j',n,n') \in \mathbb{Z}^4$$

A particular case : the dyadic case

The scales are discretized by $a = 2^j$.

$$\left\{ \psi_{j,n}(t) = \frac{1}{\sqrt{2^j}} \psi\left(\frac{t - 2^j n}{2^j}\right) \right\}_{(j,n)\in\mathbb{Z}^2}$$

Many advantages :

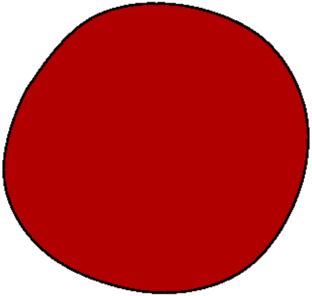
- It is “easy” to build orthogonal bases with specific properties (regularity, support compactness, . . .).
- Direct link with the theory of conjugate mirror filters.
- Fast algorithms using filter banks and subsampling.
- Easy to construct a Multi-Resolution Analysis (MRA).

Multi-Resolution Analysis (MRA)

MRA

- $\forall (j,k) \in \mathbb{Z}^2, f(t) \in V_j \Leftrightarrow f(t - 2^j k) \in V_j$
- $\forall j \in \mathbb{Z}, V_{j+1} \subset V_j$
- $\forall j \in \mathbb{Z}, f(t) \in V_j \Leftrightarrow f\left(\frac{t}{2}\right) \in V_{j+1}$
- $\lim_{j \to +\infty} V_j = \bigcap_{j=-\infty}^{+\infty} V_j = \{0\}$
- $\lim_{j \to -\infty} V_j = \bigcup_{j=-\infty}^{+\infty} V_j = L^2(\mathbb{R})$
- $\exists \{\theta(t-n)\}_{n \in \mathbb{Z}}$, Riesz basis, of V_0.

V_j



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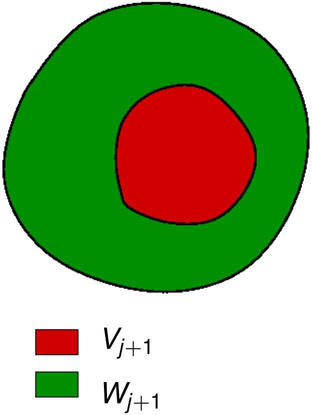

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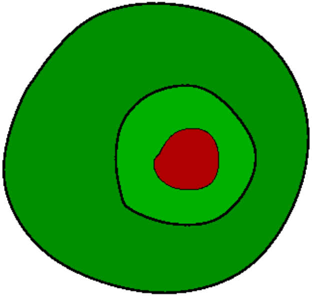

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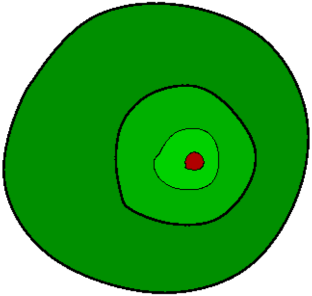

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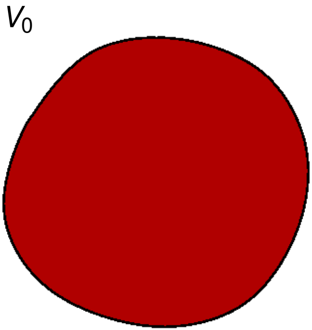

MRA and wavelets

$V_j \Leftrightarrow$ low resolutions $\Leftrightarrow a_j[n] = \langle f, \phi_{j,n} \rangle$
$W_j \Leftrightarrow$ high resolutions (details) $\Leftrightarrow d_j[n] = \langle f, \psi_{j,n} \rangle$

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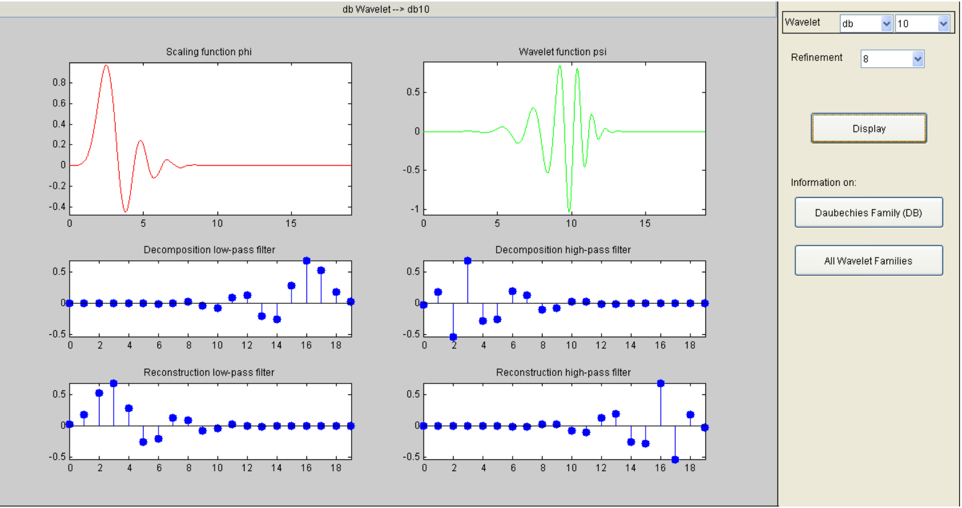
Theorem of Mallat (recursive transform)

We can build numerical filters h and g corresponding respectively to ϕ and ψ such that

$$a_{j+1}[p] = \sum_{n=-\infty}^{+\infty} h[n-2p] a_j[n]$$

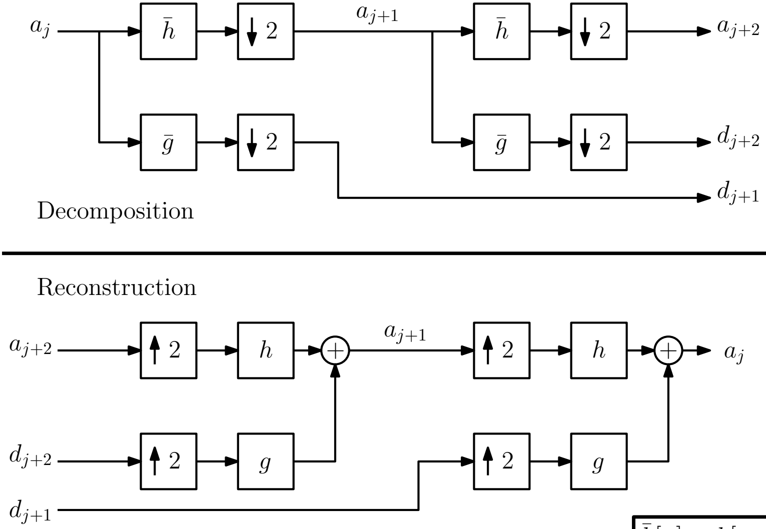
$$d_{j+1}[p] = \sum_{n=-\infty}^{+\infty} g[n-2p] a_j[n]$$

Example



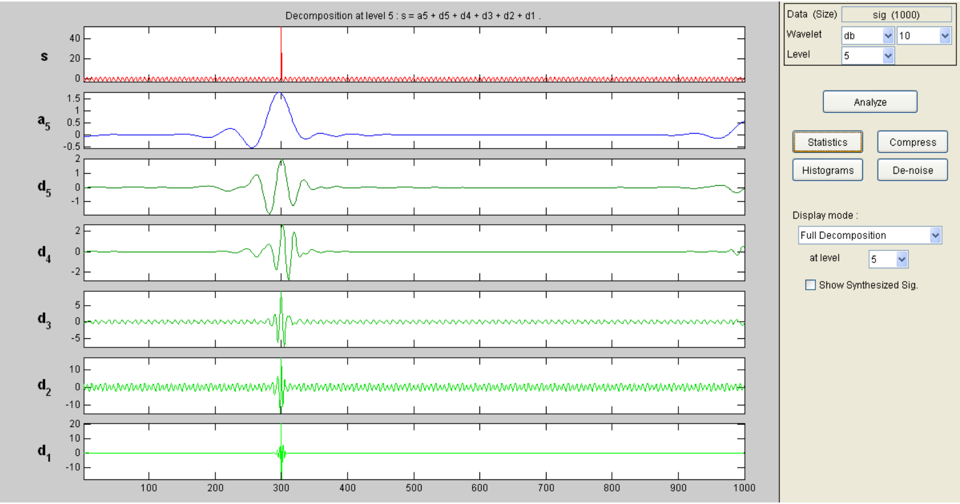
db Wavelet --> db10
Scaling function phi
Wavelet function psi
Decomposition low-pass filter
Decomposition high-pass filter
Reconstruction low-pass filter
Reconstruction high-pass filter
Wavelet
db
10
Refinement
8
Display
Information on:
Daubechies Family (DB)
All Wavelet Families

Fast Wavelet Transform



$$\bar{h}[n] = h[-n]$$
$$\bar{g}[n] = g[-n]$$

Examples (without subsampling)



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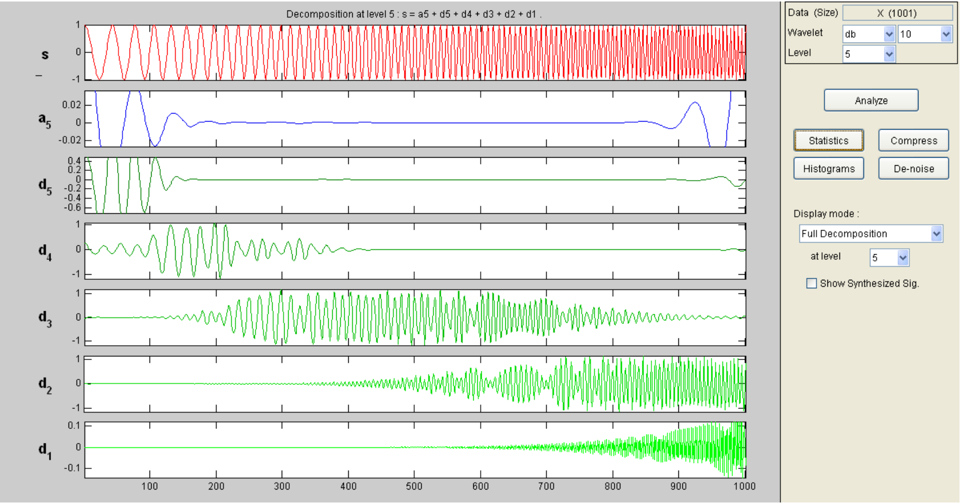
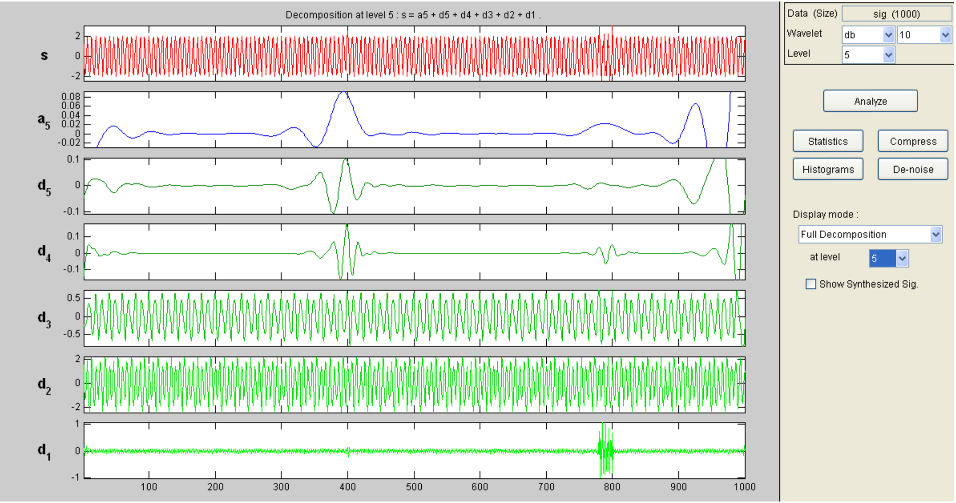


Examples (without subsampling)



2D extension

$\Rightarrow$ 2 variables $\Rightarrow$ $\phi(x, y)$ and $\psi(x, y)$.

2D extension

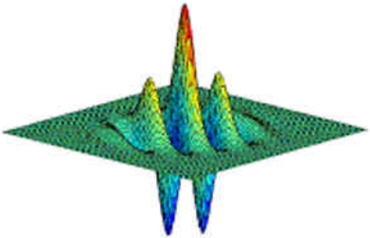
$\Rightarrow$ 2 variables $\Rightarrow \phi(x, y)$ and $\psi(x, y)$.

Two strategies

- Building directly functions of two variables (ex : Gabor filters).
- Using separable filters hence using two 1D transforms with respect to each variable.

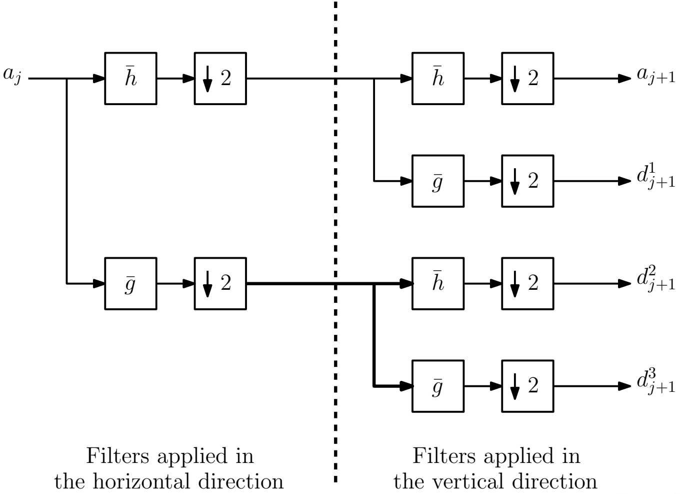
2D extension : direct construction

Ex : Gabor wavelets, Morlet's wavelet, Cauchy's wavelets, . . .

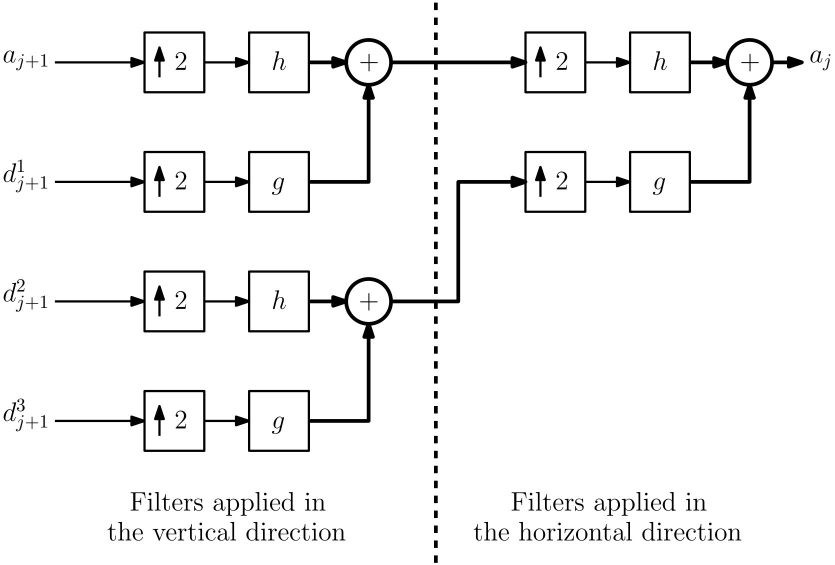


2D extension : separable transforms

Idea : we filter (1D transforms) first in one direction then in the other.
⇒ Direct extension of Mallat's algorithm.



2D extension : inverse separable transform

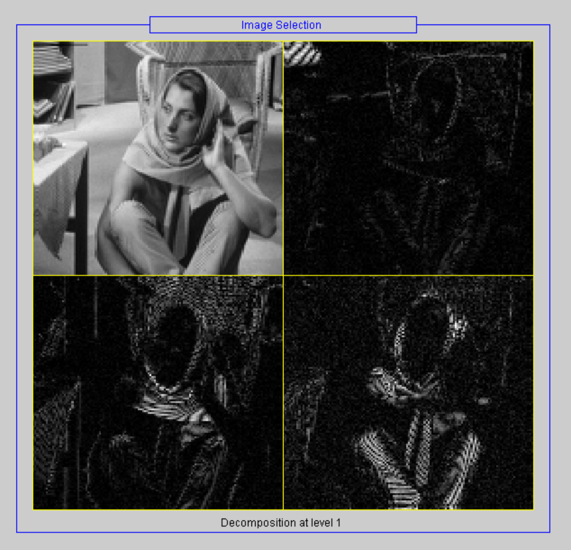


Filters applied in the vertical direction

Filters applied in the horizontal direction

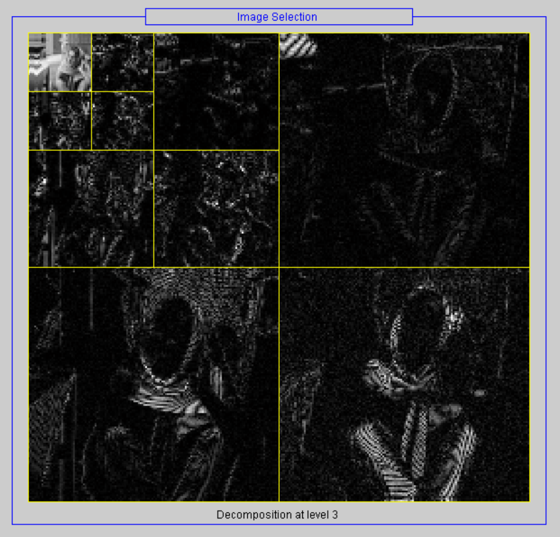
2D transform example

2D transform example

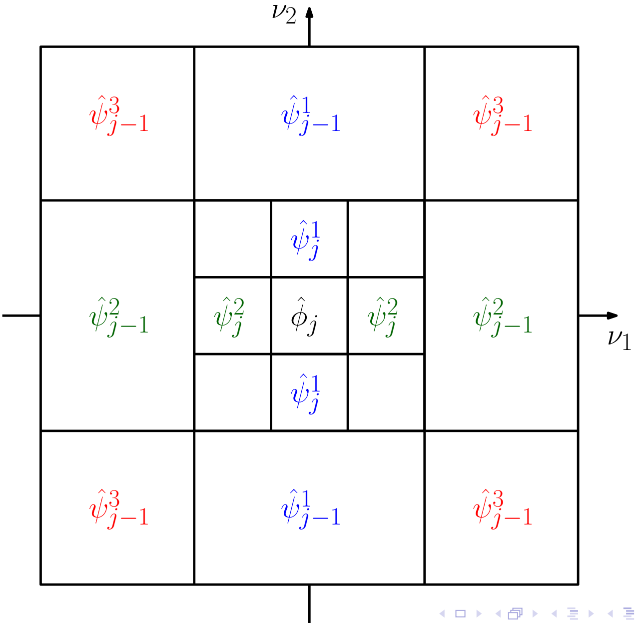


2D transform example

2D transform example



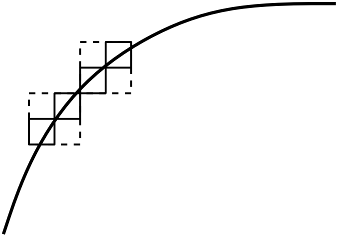
Interpretation in the Fourier domain



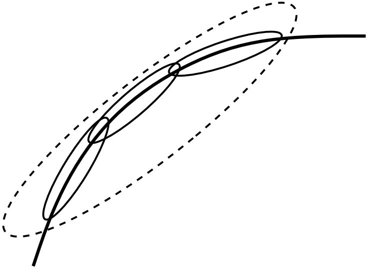
2D : a very particular world !

Separable wavelets ⇒ analysis with respect the horizontal and vertical directions.

But in an image the information can follow any direction.



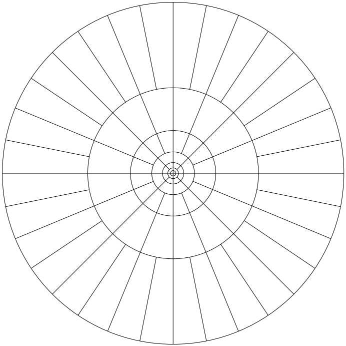
Separable wavelet approximation



Desired approximation

Better families . . .

It is possible to build frames adapted to the idea of direction, eventually to the geometry itself.



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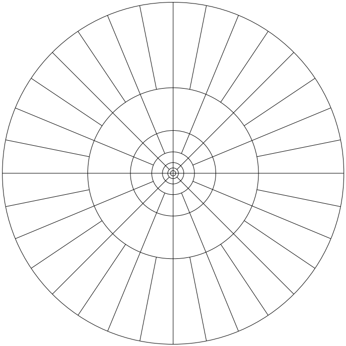
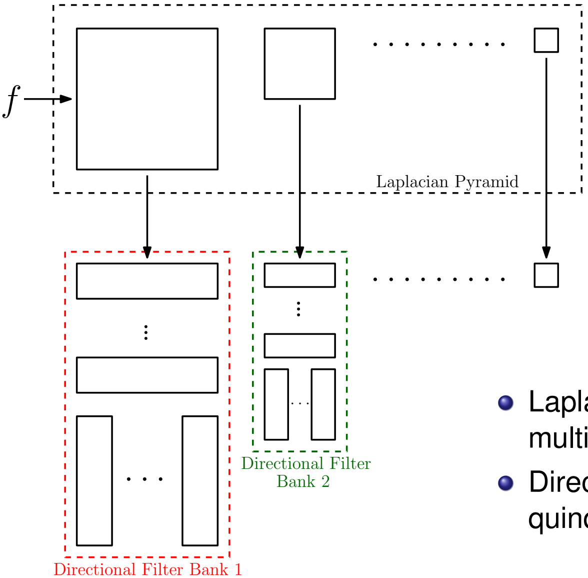


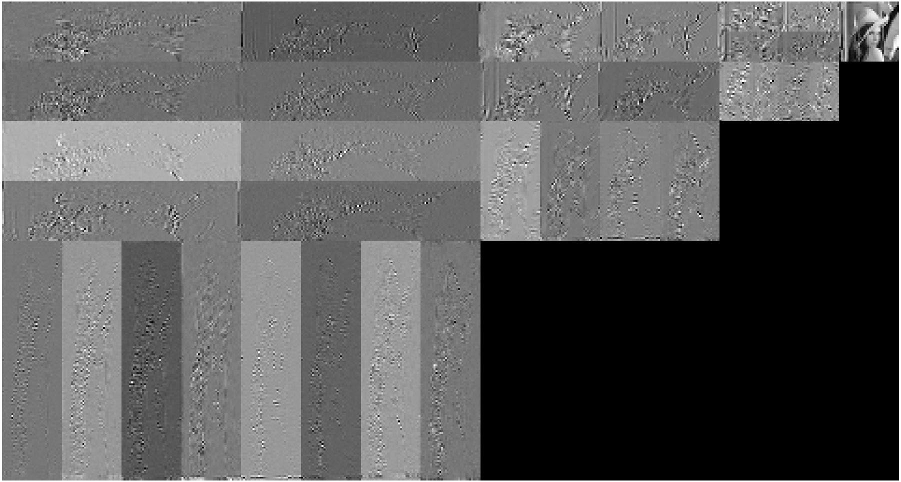
$\Rightarrow$ ridgelets, curvelets, contourlets, edgelets, bandelets,. . .

Contourlets



- Laplacian Pyramid to get the multi-resolution property
- Directional filters based on quinqux filters

Contourlets



Applications

- Denoising
- Deconvolution
- Compression
- ...

Premises : notion of approximation theory

Each coefficient contains more or less important information

Original

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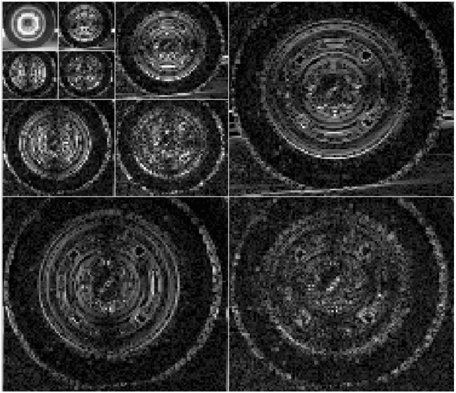
Each coefficient contains more or less important information

Original

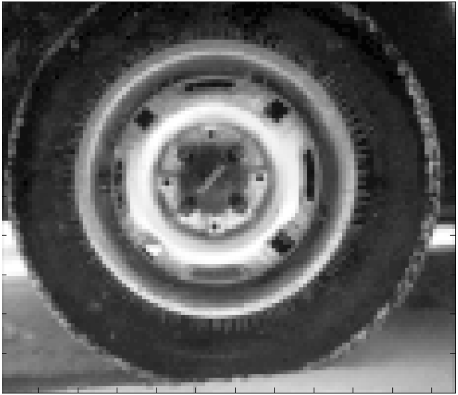


Wavelet Coefficients (Daubechies)

Premises : notion of approximation theory

Each coefficient contains more or less important information

Original



Reconstruction without the highest frequencies

Premises : notion of approximation theory

Soft thresholding

Let a threshold T

$$HT(x,T) = \begin{cases} 0 & \text{if} \quad |x| \leqslant T \\ sign(x)(|x| - T) & \text{if} \quad |x| > T \end{cases}$$

Hard thresholding)

Let a threshold T

$$HT(x,T) = \begin{cases} 0 & \text{if} \quad |x| \leqslant T \\ x & \text{if} \quad |x| > T \end{cases}$$

Example of soft thresholding

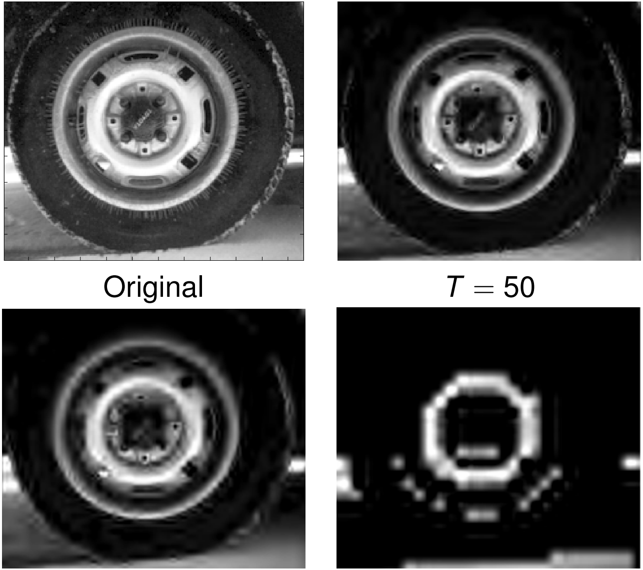
Original

$T = 50$

$T = 100$

$T = 1000$

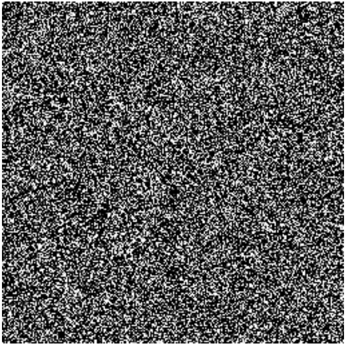
Example of hard thresholding



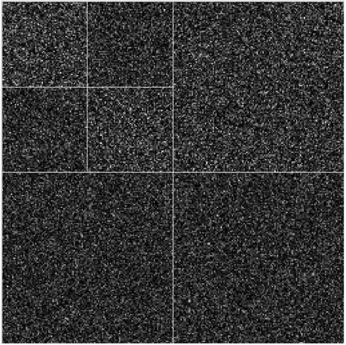
Original $\quad T = 50$

$T = 100 \quad T = 1000$

Denoising



Noise

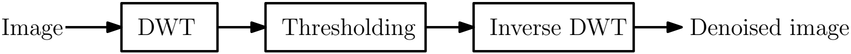


Wavelet coefficients of noise

The noise energy is distributed among all scales (low amplitude coefficients).

Denoising

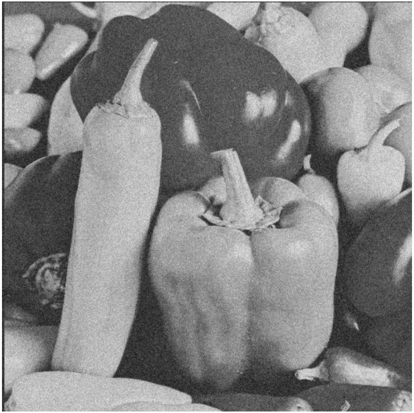
⇒ we can use some thresholding to remove the noise coefficients.



Gaussian noise ⇒ soft thresholding is optimal in theory but hard thresholding provides visually better results.

Denoising : example

Original



Noisy Version

Denoising : example

Original

Soft thresh. on wavelet coefs

Denoising : example

Original

Hard thresh. on wavelet coefs

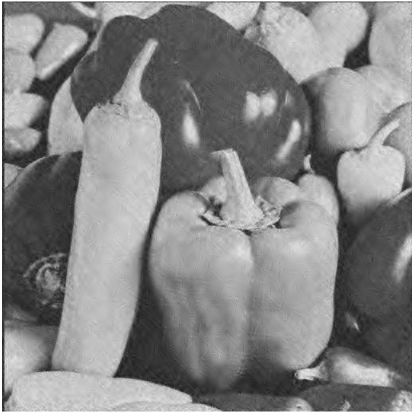
Denoising : example

Original

Soft thresh. on contourlet coefs

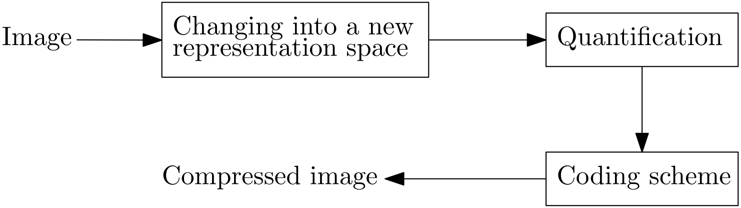
Denoising : example

Original

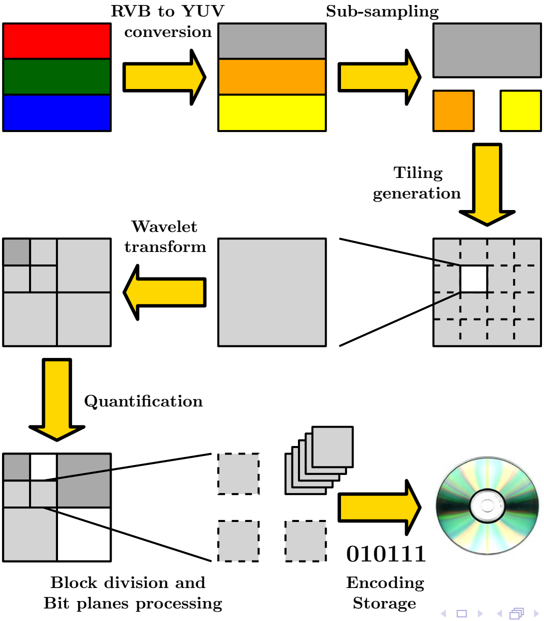


Hard thresh. on contourlet coefs

Compression : general principle



Compression : JPEG2000



Compression : JPEG2000

JPEG 1 :86

JPEG 1 :41

JPEG2000 1 :86

JPEG2000 1 :41

Wavelet Wonderland . . .

From a theoretical point of view :

- Other extensions : wavelet packets, rational wavelets, . . .
- Useful tool in functional analysis (Besov spaces, Triebel-Lizorkin spaces), . . .
- Direct link with the approximation theory, . . .
- Useful tool to solve differential equations, . . .

Wavelet Wonderland . . .

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- Useful tool to solve differential equations, . . .

From the application point of view :

- Video Compression (MPEG4, . . .)
- Signal analysis : seismic, acoustic, . . .
- Image processing : texture analysis, modeling, . . .
- . . .

Some references

- S.Mallat, “A Wavelet Tour of Signal Processing, 3 Ed”
- M.Vetterli, “Wavelets and subband coding” (http://infoscience.epfl.ch/record/33934/files/VetterliKovacevic95_Manuscript.pdf?version=1)
- Y.Meyer, “Wavelets and operators” (3 volumes)
- Wavelet Digest : http://www.wavelet.org